i.e. one considers a large number of identical systems and takes the average of the velocity at corresponding instants over all these systems. In practice, the average is usually a time average; one observes and averages the velocity at a point over a period long enough for separate measurements to give effectively the same result. Procedural difficulties can arise when the imposed conditions are unsteady, but we need not consider such situations here.

Thus throughout the following the average of any quantity Q signifies

$$\bar{Q} = \frac{1}{2s}\int_{-s}^{s} Q\, dt \tag{19.1}$$

where s is large compared with any of the time scales involved in the variations of Q.

U indicates the mean motion of the fluid. Information about the structure of the velocity fluctuations is given by other average quantities, the first being the mean square fluctuation, $\overline{u^2}$. $(\overline{u^2})^{1/2}$ is known as the intensity of the turbulence component, and

$$(\overline{q^2})^{1/2} = (\overline{u^2} + \overline{v^2} + \overline{w^2})^{1/2} \tag{19.2}$$

as the intensity of the turbulence. It is directly related to the kinetic energy per unit volume associated with the velocity fluctuations,

$$\Sigma = \tfrac{1}{2}\rho\overline{q^2}. \tag{19.3}$$

The same intensity field can in principle be produced by many different patterns of velocity fluctuation. Before we look at the average quantities most often used to examine the more detailed structure of the velocity field, we consider briefly an alternative representation. This is in a sense the most fundamental statistical representation, although it is not the most convenient for the development of models of turbulent structure based on experimental observation. The probability distribution function $P(u)$ of a velocity component at one point is defined so that the probability that the fluctuation velocity is between u and $u + du$ is $P(u)\,du$. One thus has

$$\int_{-\infty}^{\infty} P(u)\, du = 1. \tag{19.4}$$

The intensity is related to this,

$$\overline{u^2} = \int_{-\infty}^{\infty} u^2 P(u)\, du, \tag{19.5}$$

but the probability distribution function contains more information than the intensity. Relationships between velocity fluctuations at different points (or times) are indicated by joint probability distribution functions.

For example a second-order function, $P(u_1, u_2)$, may be defined so that the probability that the velocity at one point lies between u_1 and $u_1 + du_1$ and that at the other point simultaneously lies between u_2 and $u_2 + du_2$ is $P(u_1, u_2)\, du_1\, du_2$. In principle, for a complete representation of the turbulence, this process has to be continued to all orders.

Probability distribution functions are sometimes determined experimentally, but much more frequently further average quantities are measured. Fuller information than is given by $\overline{u^2}$ about the fluctuations at a single point can be obtained by measurements of $\overline{u^3}$, $\overline{u^4}$, etc.

Information about velocity fluctuations at different points (or times) is given by correlation measurements. The correlation between two velocity fluctuations u_1 and u_2 is defined as $\overline{u_1 u_2}$ and the correlation coefficient as

$$R = \overline{u_1 u_2}/(\overline{u_1^2}\,\overline{u_2^2})^{1/2}. \tag{19.6}$$

In this definition u_1 and u_2 are quite general quantities; but as examples, they could be simultaneous values of the same component of the velocity at two different points, or two different components of the velocity at a single point. If the fluctuations u_1 and u_2 are quite independent of one another, then their correlation is zero. However, any turbulent flow is governed by the usual equations and these do not allow such complete independence, particularly for fluctuations at points close to one another. Hence, non-zero correlations are observed.

The concept of correlations, like that of probability distributions, can be extended to higher orders, by defining quantities such as $\overline{u_1 u_2 u_3}$. A complete specification of the turbulence again requires one to consider all orders up to infinity. In practice, detailed attention is usually confined to double correlations ($\overline{u_1 u_2}$) with briefer investigation of triple correlations.

Experimental studies of turbulent flows often involve the interpretation of correlation measurements. One of the reasons for working particularly with correlations is that those of low order lend themselves satisfactorily to physical interpretation, in ways to be discussed in Section 19.4. We shall also be introducing later (Section 19.5) the spectrum functions which are the Fourier transforms of correlation functions. However, we now have enough material to examine the way in which the equations of motion are developed for turbulent flows.

19.3 Turbulence equations

In the interests of conciseness and convention, it is necessary to use here the suffix notation for vector equations which has for the most part been avoided in this book (the other main exception being the appendix to Chapter 5). For a full explanation of this notation see Refs. [26, 44, 212].

Its basic features are as follows. Each suffix can take values 1, 2 or 3, corresponding to the three coordinate directions; a vector equation can be read as any one of its component equations by substituting the appropriate value for the suffix common to every term; and the repetition of a suffix within a single term indicates that that term is summed over the three values of that suffix.

With the velocity divided into its mean and fluctuating parts, the continuity equation (5.10) is

$$\operatorname{div}(\boldsymbol{U}+\boldsymbol{u})=0 \tag{19.7}$$

that is

$$\partial(U_i+u_i)/\partial x_i=0. \tag{19.8}$$

Averaging this equation (the processes of averaging and differentiation are interchangeable in order),

$$\partial U_i/\partial x_i=0. \tag{19.9}$$

Subtracting this result from the original equation, we have

$$\partial u_i/\partial x_i=0. \tag{19.10}$$

The mean and fluctuating parts of the velocity field thus individually satisfy the usual form of the continuity equation.

The same division applied to the Navier–Stokes equation (eqn (5.22) with $\boldsymbol{F}=0$) gives

$$\frac{\partial(U_i+u_i)}{\partial t}+(U_j+u_j)\frac{\partial(U_i+u_i)}{\partial x_j}=-\frac{1}{\rho}\frac{\partial(P+p)}{\partial x_i}+\nu\frac{\partial^2(U_i+u_i)}{\partial x_j^2}. \tag{19.11}$$

Carrying out the averaging process throughout this equation gives

$$\frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial x_j}+\overline{u_j\frac{\partial u_i}{\partial x_j}}=-\frac{1}{\rho}\frac{\partial P}{\partial x_i}+\nu\frac{\partial^2 U_i}{\partial x_j^2} \tag{19.12}$$

which, with the aid of the continuity equation (19.10), may be rewritten

$$U_j\frac{\partial U_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial P}{\partial x_i}+\nu\frac{\partial^2 U_i}{\partial x_j^2}-\frac{\partial}{\partial x_j}(\overline{u_iu_j}) \tag{19.13}$$

where, additionally, attention has been restricted to steady mean conditions by making the first term of (19.12) zero.

Equation (19.13) for the mean velocity U_i differs from the laminar flow equation by the addition of the last term. This term represents the action of the velocity fluctuations on the mean flow arising from the non-linearity of the Navier–Stokes equation. It is frequently large compared with the viscous term, with the result that the mean velocity distribution is very different from the corresponding laminar flow.

The character of this interaction between the mean flow and the fluctuations can be seen most simply in the context of a flow for which the two-dimensional boundary layer approximation applies. The turbulent fluctuations are always three-dimensional, but if the imposed conditions are two-dimensional, there is no variation of mean quantities in the third direction and terms such as $\partial(\overline{uw})/\partial z$ (that would otherwise appear in the next equation) are zero. Omitting such terms and terms that are small on the boundary layer approximation† in eqn (19.13) gives the turbulent flow counterpart of eqn (11.8); that is

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\frac{\partial^2 U}{\partial y^2}-\frac{\partial}{\partial y}(\overline{uv}). \tag{19.14}$$

This equation is applied to turbulent boundary layers, jets, wakes, etc.

Writing the last two terms of (19.14) as

$$\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial U}{\partial y}-\rho\overline{uv}\right) \tag{19.15}$$

shows that the velocity fluctuations produce a stress on the mean flow. A gradient of this produces a net acceleration of the fluid in the same way as a gradient of the viscous stress. The quantity $(-\rho\overline{uv})$, and more generally the quantity $(-\rho\overline{u_iu_j})$, is called a Reynolds stress.

The Reynolds stress arises from the correlation of two components of the velocity fluctuation at the same point. A non-zero value of this correlation implies that the two components are not independent of one another. For example, if $\overline{uv}$ is negative, then at moments at which u is positive, v is more likely to be negative than positive; conversely when u is negative. Transferring attention to coordinates at 45° to the x- and y-directions shows that this corresponds to anisotropy of the turbulence—different intensities in different directions. Putting

$$u'=(u+v)/\sqrt{2} \qquad v'=(v-u)/\sqrt{2} \tag{19.16}$$

gives

$$\overline{uv}=\tfrac{1}{2}(\overline{u'^2}-\overline{v'^2}). \tag{19.17}$$

Figure 19.2 shows the geometrical significance of this.

One can readily see how a correlation of this kind can arise in a mean shear flow. We may consider the case of positive $\partial U/\partial y$ as shown in Fig. 19.3. A fluid particle with positive v is being carried by the turbulence in

† The boundary layer approximation is used here to its fullest extent. In some studies of turbulent flows, some further terms (e.g. $\partial(\overline{u^2})/\partial x$) are retained because measurements indicate that they are not so very small.

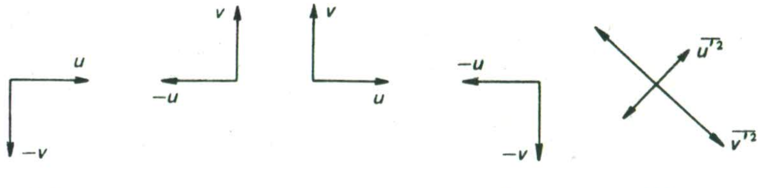

FIG. 19.2 Geometrical interpretation of Reynolds stress: if patterns of velocity fluctuations shown in (a) and (b) occur more frequently than those in (c) and (d), giving negative $\overline{uv}$, then $\overline{v'^2}$ is larger than $\overline{u'^2}$ as indicated by (e).

the positive y-direction. It is coming from a region where the mean velocity is smaller and it is thus likely to be moving downstream more slowly than its new environment; i.e. it is more likely to have negative u than positive. Similarly negative v is more likely to be associated with positive u. The process is in general (but not in detail) analogous to the Brownian motion of molecules giving rise to fluid viscosity.

The analogy has led to the definition of a quantity ν_T such that

$$-\overline{uv}=\nu_T\partial U/\partial y \qquad (19.18)$$

ν_T is called the eddy viscosity. It is important to realize that ν_T is a representation of the action of the turbulence on the mean flow and not a property of the fluid. It is moreover a representation that simplifies the dynamics of that action; because of the large-scale coherent motions (Sections 21.4, 21.6), the Reynolds stress at any point depends on the whole velocity profile, not just the local gradient. Although it is sometimes useful for approximate calculations to suppose that ν_T is an (empirical) constant, in general (19.18) should be regarded as the defining equation of ν_T rather than an equation for $\overline{uv}$.

Further understanding of the interaction between the mean flow and the fluctuations is obtained from the equation for the kinetic energy of

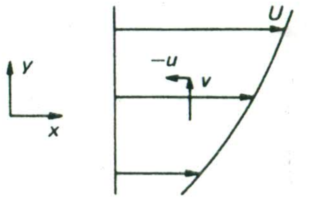

FIG. 19.3 To illustrate the generation of a Reynolds stress in a mean velocity gradient.

the turbulence. Subtracting eqn (19.12) from eqn (19.11) gives

$$\frac{\partial u_i}{\partial t}+U_j\frac{\partial u_i}{\partial x_j}+u_j\frac{\partial U_i}{\partial x_j}+u_j\frac{\partial u_i}{\partial x_j}-\overline{u_j\frac{\partial u_i}{\partial x_j}}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_j^2}. \qquad (19.19)$$

Multiplying this by u_i and averaging

$$\frac{1}{2}\frac{\partial(\overline{u_i^2})}{\partial t}+\frac{1}{2}U_j\frac{\partial(\overline{u_i^2})}{\partial x_j}=-\overline{u_iu_j}\frac{\partial U_i}{\partial x_j}-\frac{1}{2}\frac{\partial}{\partial x_j}(\overline{u_i^2u_j})-\frac{1}{\rho}\frac{\partial}{\partial x_i}(\overline{pu_i})+\nu\overline{u_i\frac{\partial^2 u_i}{\partial x_j^2}}$$

(19.20)

(where the rearrangement of terms has made use of the continuity equation (19.10)). Since the summation convention is being applied, the mathematics involves multiplying each component of the dynamical equation (19.19) by the corresponding velocity component and then adding the three resulting equations. For steady mean conditions the first term of eqn (19.20) is zero, but it indicates the physical significance of the equation; in view of the summation convention

$$\overline{u_i^2}=\overline{q^2}=2\Sigma/\rho \qquad (19.21)$$

and so each term in the equation represents some process tending to increase or decrease the kinetic energy of the turbulence.

With the boundary layer approximation applied to a flow which is steady and two-dimensional in the mean, eqn (19.20) becomes

$$\frac{1}{2}U\frac{\partial(\overline{q^2})}{\partial x}+\frac{1}{2}V\frac{\partial(\overline{q^2})}{\partial y}=-\overline{uv}\frac{\partial U}{\partial y}-\frac{\partial}{\partial y}\left(\frac{1}{2}\overline{q^2v}+\frac{1}{\rho}\overline{pv}\right)+\nu\overline{u_i\frac{\partial^2 u_i}{\partial x_j^2}}. \qquad (19.22)$$

The left-hand side and the second term on the right-hand side are terms that become zero when integrated over the whole flow. They represent the transfer of energy from place to place, respectively transfer by the mean motion and transfer by the turbulence itself. As in a laminar flow (Section 11.7), the viscous term can be divided into two parts: one is essentially negative and thus represents viscous dissipation; the other (usually small) integrates to zero and so is another energy transfer process. The input of energy to compensate for the dissipation must be provided by the only remaining term, $(-\overline{uv}\,\partial U/\partial y)$. We have seen that $\overline{uv}$ is likely to be negative where $\partial U/\partial y$ is positive, giving this term the required sign. Although local regions of positive $(\overline{uv}\,\partial U/\partial y)$ can occur, they cannot occupy the majority of the flow or the turbulence cannot be maintained.†

† This statement need not be true for systems described by a dynamical equation with additional terms to those in (19.11); for example a buoyancy force. These can give further terms in the energy equation which may represent alternative turbulence-generating mechanisms.

The equation for the energy of the mean flow contains a corresponding term of opposite sign. The term thus represents a transfer of energy from the mean flow to the turbulence. One can therefore say that the Reynolds stress works against the mean velocity gradient to remove energy from the mean flow, just as the viscous stress works against the velocity gradient. However, the energy removed by the latter process is directly dissipated, reappearing as heat, whereas the action of the Reynolds stress provides energy for the turbulence. This energy is ultimately dissipated by the action of viscosity on the turbulent fluctuations. Frequently, the loss of mean flow energy to turbulence is large compared with the direct viscous dissipation.

19.4 Interpretation of correlations

Correlation coefficients (Section 19.2) play an important role in both theoretical and experimental studies of turbulence. To illustrate how they can indicate the scale and structure of a turbulent motion, we now look at typical properties of double correlations. Some of the ideas introduced rather vaguely here will be used more specifically in Sections 20.2, 21.4 and 21.6.

When u_1 and u_2 are velocities at different positions but the same instant, $\overline{u_1 u_2}$ is known as a space correlation. Its particulars may be specified by a diagram such as Fig. 19.4(a). Most attention is usually given to correlations of the same component of velocity at points separated in a direction either parallel to that velocity component (Fig. 19.4(b)) or perpendicular to it (Fig. 19.4(c)). We may call these respectively longitudinal and lateral correlations.

The correlation will depend on both the magnitude and direction of the separation, $\boldsymbol{r}$. Different behaviours in different directions may provide information about the structure of the turbulence, a point that will be taken up again in Section 21.4. Here we pay more attention to the variation with distance, $r = |\boldsymbol{r}|$. When $r = 0$, $u_1 = u_2$ (provided they are in the same direction) and the correlation coefficient R of eqn (19.6) is, by definition, equal to 1. At large r the velocity fluctuations become independent of one another and R asymptotically approaches 0. In consequence, the dependence of R on r takes typically one of the forms shown in Fig. 19.5. (r has a maximum value of 1 at $r = 0$ and so $(\partial R/\partial r)_{r=0} = 0$. However, the curvature at $r = 0$ is usually large and experimentally measured correlations often appear to have finite slope at $r = 0$.)

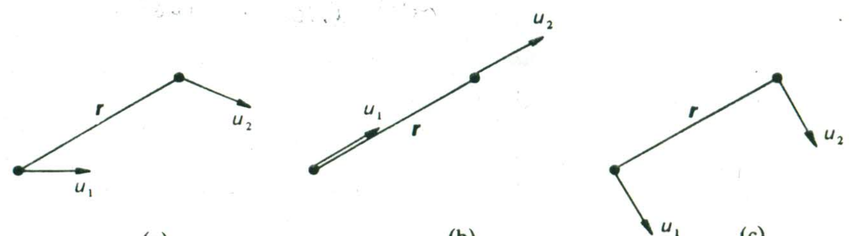

FIG. 19.4 Schematic representation of double velocity correlations.

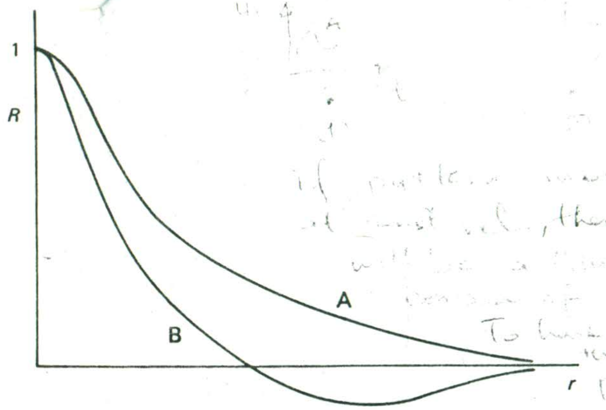

FIG. 19.5 Typical correlation curves.

A negative region in the correlation curve implies that u_1 and u_2 tend to be in opposite directions more than in the same direction. For a longitudinal correlation this would imply dominant converging and/or diverging flow patterns. There is often no reason to expect such patterns and one may expect that longitudinal correlations will usually give a curve such as A (Fig. 19.5). On the other hand, lateral correlations may be expected to have a negative region, like curve B, since continuity requires the instantaneous transport of fluid across any plane (by the fluctuations) to be zero. Such expectations are not always fulfilled; but, when they fail, this may itself be informative about the structure of the turbulence.

A correlation curve indicates the distance over which the motion at one point significantly affects that at another. It may be used to assign a length scale to the turbulence; a length can be defined for example as $\int_0^\infty R\,dr$, or as the distance in which R falls to $1/e$, or, if the curve has a negative region, the value of r at which R is a minimum. We shall see in Sections 19.6, 20.3, and 21.3 that this concept is extended to associate a variety of length scales with the turbulence.

The correlation of the same velocity component at a single point at different instants is known as an autocorrelation. Such a correlation