VECTOR IDENTITIES

In the following formulas, ϕ is any scalar and **a**, **b**, and **c** are any vectors.

$$\nabla \times \nabla\phi = 0$$

$$\nabla \cdot (\phi\mathbf{a}) = \phi\nabla \cdot \mathbf{a} + \mathbf{a} \cdot \nabla\phi$$

$$\nabla \times (\phi\mathbf{a}) = \nabla\phi \times \mathbf{a} + \phi(\nabla \times \mathbf{a})$$

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0$$

$$(\mathbf{a} \cdot \nabla)\mathbf{a} = \tfrac{1}{2}\nabla(\mathbf{a} \cdot \mathbf{a}) - \mathbf{a} \times (\nabla \times \mathbf{a})$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a}$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$

INTEGRAL THEOREMS

In the following two theorems, which relate surface integrals to volume integrals, V is any volume and S is the surface which encloses V, the unit normal on S being denoted by **n**. ϕ is any scalar and **a** is any vector.

Gauss' theorem (also known as the divergence theorem):

$$\int_s \mathbf{a} \cdot \mathbf{n}\, ds = \int_V \nabla \cdot \mathbf{a}\, dV$$

Green's theorem:

$$\int_s \phi \frac{\partial\phi}{\partial n}\, dS = \int_V [\nabla\phi \cdot \nabla\phi + \phi\nabla^2\phi] dV$$

Stokes' theorem:

$$\oint_l \mathbf{a} \cdot \mathbf{dl} = \int_A (\nabla \times \mathbf{a}) \cdot \mathbf{n}\, dA$$

This theorem relates a line integral to an equivalent surface integral. The surface A is arbitrary, but it must terminate on the line **l**.

Notation:

The following rules of notation will be followed throughout:

1 If a given index appears only once in each term of a tensor equation, it is a free index and the equation holds for all possible values of that index.

2 If an index appears twice in any term, it is understood that a summation is to be made over all possible values of that index.

3 No index may appear more than twice in any term.

Definition:

A tensor of rank r is a quantity having n^r components in n-dimensional space. The components of a tensor quantity expressed in two different coordinate systems are related as follows:

$$T'_{ijk\cdots m} = C_{is}C_{jt}C_{ku}\cdots C_{mv}T_{stu\cdots v}$$

where the quantities C_{mn} are the direction cosines between the axes of the two coordinate systems.

A tensor of rank 2 is sometimes called a *dyadic*, a tensor of rank 1 is a *vector*, and a tensor of rank 0 is a *scalar*.

TENSOR ALGEBRA

Addition:

Two tensors of equal rank may be added to yield a third tensor of the same rank as follows:

$$C_{ij\cdots k} = A_{ij\cdots k} + B_{ij\cdots k}$$

Multiplication:

If tensor A has rank a and tensor B has rank b, the multiplication of these two tensors yields a third one of rank c.

$$C_{ij\cdots krs\cdots t} = A_{ij\cdots k}B_{rs\cdots t}$$

Contraction:

If any two indices of a tensor of rank $r \geq 2$ are set equal, a tensor of rank $r-2$ is obtained. For example, if C_{ij} is defined by

$$C_{ij} = A_iB_j$$

then by setting $i=j$ the tensor C_{ij}, which is of rank 2, becomes a tensor of rank 0 (i.e., a scalar).

$$C_{ii} = A_iB_i$$

Thus contraction is equivalent to taking the scalar product of two vectors in vector algebra.

Symmetry:

If the tensor A has the property that

$$A_{i\cdots j\cdots k\cdots l} = A_{i\cdots k\cdots j\cdots l}$$

then the tensor A is said to be *symmetric* in the indices j and k. As a consequence of the above relation the tensor has only $\frac{1}{2}n(n+1)$ independent components.

If the tensor A has the property that

$$A_{i\cdots j\cdots k\cdots l} = -A_{i\cdots k\cdots j\cdots l}$$

then the tensor A is said to be *antisymmetric* in the indices j and k. Such tensors have only $\frac{1}{2}n(n-1)$ independent components.

TENSOR OPERATIONS

Gradient:

The gradient of a tensor of rank r is defined by

$$T_{ij\cdots kl} = \frac{\partial R_{ij\cdots k}}{\partial x_l}$$

and yields a tensor of rank $(r+1)$.

Divergence:

The divergence of a tensor of rank r results in a tensor of rank $(r-1)$.

$$T_{i\cdots jl\cdots m} = \frac{\partial R_{i\cdots jkl\cdots m}}{\partial x_k}$$

Curl:

If R is a tensor of rank r, the curl operation will produce an antisymmetric tensor of rank $(r+1)$. In general, the operation is defined by

$$T_{i\cdots j\cdots kl} = \frac{\partial R_{i\cdots j\cdots k}}{\partial x_l} - \frac{\partial R_{i\cdots l\cdots k}}{\partial x_j}$$

In three dimensions, the curl of a tensor of rank 1 (i.e., a vector) may be written in the form

$$T_i = -\varepsilon_{ijk}\frac{\partial R_j}{\partial x_k}$$

where ε_{ijk} is a constant pseudoscalar defined by

$$\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$$
$$\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$$
$$\varepsilon_{ijk} = 0 \text{ otherwise}$$

ISOTROPIC TENSORS

Definition:

An isotropic tensor is one whose components are invariant with respect to all possible rotations of the coordinate system. That is, there are no preferred directions, and the quantity represented by the tensor is a function of position only and not of orientation.

Isotropic tensors of rank 0:

All tensors of rank 0 (i.e., scalars) are isotropic.

Isotropic tensors of rank 1:

There are no isotropic tensors of rank 1. That is, vectors are not isotropic, since there are preferred directions.

Isotropic tensors of rank 2:

The only isotropic tensors of rank 2 are of the form $\alpha\delta_{ij}$, where α is a scalar and δ_{ij} is the *Kronecker delta*, which has the property that

$$\delta_{ij} = \begin{cases} 0 \text{ when } & i \neq j \\ 1 \text{ when } & i = j \end{cases}$$

Isotropic tensors of rank 3:

The isotropic tensors of rank 3 are of the form $\alpha\varepsilon_{ijk}$, where α is a scalar and ε_{ijk} is a pseudoscalar defined under Tensor Operations.

Isotropic tensors of rank 4:

The most general isotropic tensor of rank 4 is of the form

$$\alpha\delta_{ij}\delta_{pq} + \beta(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}) + \gamma(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})$$

where α, β, and γ are scalars.

INTEGRAL THEOREMS

The following two theorems were given in vector form in Appendix A, and they are reproduced here in tensor form.

Gauss' theorem (divergence theorem):

$$\int_s a_i n_i \, dS = \int_V \frac{\partial a_i}{\partial x_i} \, dV$$

Stokes' theorem:

$$\oint a_i \, dl_i = \int_A - \varepsilon_{ijk} \frac{\partial a_j}{\partial x_k} n_i \, dA$$